Stresses and conjugate strain-increments in plotting experimental data for unsaturated soils

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ABSTRACT: The development of an equation describing the stress regime in an unsaturated soil is presented. The analysis is contrasted with Bishop's (1959) modified effective stress equation for unsaturated soils and a more recent analysis by Houlsby (1997) of the work input to an unsaturated soil. The equation is presented in terms of the stress state variables ($\sigma_{ij} - u_a\,\delta_{ij}$) and ($u_a - u_w$) along with the volumes through which the stresses act. The analysis is extended to triaxial stress conditions and to the development of work input equations that give the work conjugate strain-increments. The conjugate strain-increments are those within the volumes through which the stress state variables act. The development of a stress-strain model and constitutive relations from experimental data requires careful consideration to be given to the choice of the stress state variables and the strain-increments if they are to be work conjugate. Plots of triaxial experimental results for a kaolin specimen are presented and illustrate the use of the conjugate variables.

1. INTRODUCTION

Bishop (1959) suggested that the behaviour of an unsaturated soil can be described in terms of a single effective stress which may be written in tensor form as,

$$\sigma'_{ij} = \sigma_{ij} - (u_a - \chi\,(u_a - u_w))\delta_{ij} \qquad (1)$$

where, σ'_{ij} is Bishop's stress tensor; σ_{ij} is the total stress tensor; u_a is pore air pressure; u_w is pore water pressure; χ is a variable often associated with the degree of saturation; and δ_{ij} is the Kronecker delta

The analysis by Houlsby (1997) into the work input to an unsaturated granular material subject to simultaneous deformation and fluid flow resulted in a form of Bishop's equation with $\chi = S_r$ (where S_r is the degree of saturation). The analysis relied on the following equilibrium equation for the total stress in an unsaturated soil,

$$\sigma_{ij} = n[S_r\,u_w + (1\text{-}S_r)\,u_a]\delta_{ij} + (1\text{-}n)s_{ij} + T_{ij} \qquad (2)$$

where, n is the porosity; s_{ij} is the average stress in the soil grains; and T_{ij} represents the forces in the contractile skin

This 'energy formulation' contains terms for the water pressure, the air pressure and the average stress in the soil grains. These are defined as acting within the volumes of air, water and solid particles respectively. There is also a term for the surface tension but the term is later dropped. However, in restricting s_{ij} to acting within the solid phase only, s_{ij} is essentially defined as a further fluid pressure and there is no term for the stress interaction between the soil particles, which is normally defined as a continuum mechanics variable associated with total planar area or volume. As a consequence Equation (2) does not reduce to Terzaghi's effective stress under saturated conditions and the significance of this is discussed in the paper.

A further point of concern is the use of the term S_r in place of χ in the Bishop equation. This is considered a poor parameter in describing the stress regime in an unsaturated soil. It defines the volume of the voids filled with water, which is likely to be important for considerations of fluid flow but does not account for the volume of the solid phase. It is the interaction of the particles making up the solid phase that gives a soil its strength and controls deformation behaviour. The omission of a term representing the volume of the solid phase is considered important, as is the affinity of the water phase to interact with the solid particles.

In developing the analysis for unsaturated soil it is considered instructive to first consider the special case of a saturated soil comprising the phases of water and soil particles and then to expand the arguments to examine the influence of introducing an air phase.

2. SATURATED SOIL

Traditionally, in explaining effective stress in saturated soils, a plane is cut through a particulate material such that the plane passes through the points of contact between particles. This has the effect of ensuring that the water pressure acts over the total planar area if the total area of the contact points between particles is assumed negligible. However, any flat plane will pass through soil particles. The representation of a plane through the contact points serves to mask the influence of the water surrounding the particles. Consider Figure 1, which shows the simple case of a horizontal plane cut through a saturated soil and in particular through a solid particle.

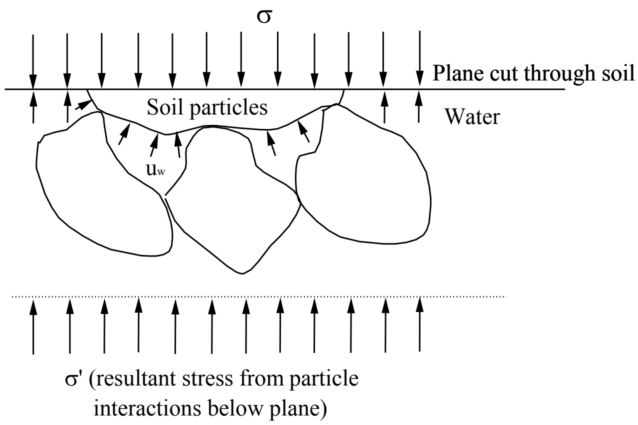

Figure 1 Plane cut through an idealized saturated soil

For equilibrium across the plane, in addition to the water pressure acting over the cut area of water, the water pressure (or buoyancy effect) acting on the area of the cut particle must also be taken into account. It is also necessary to account for the effective stress σ', this being a simplification of the actual stress conditions as a result of particle interactions but is a necessary generalisation applicable to continuum mechanics. In the representation in Figure 1, σ' acts below the plane as drawn and is resolved as a net reaction in the vertical direction. By definition, in continuum mechanics the effective stress acts over the total planar area. If the total applied stress σ acts on the upper part of the plane of area A then for equilibrium,

$$\sigma A = \sigma' A + u_w (A_w + A_s) \qquad (3)$$

where, A_w is the area of water cut by the plane and A_s is the area of solid cut by the plane.

Since for a saturated soil $A = A_w + A_s$, Terzaghi's equation emerges,

$$\sigma = \sigma' + u_w \qquad (4)$$

Terzaghi's equation may be written in stress tensor form as,

$$\sigma_{ij} = \sigma'_{ij} + u_w \, \delta_{ij} \qquad (5)$$

There are no area terms in equations (4) and (5) as the area terms in equation (3) cancel out. Adopting an alternative 'energy formulation' for a saturated soil, it is necessary to take account of the following:

- The water pressure acting through the volume of water;
- The water pressure acting through the volume of solids as the water surrounds the particles giving rise to a buoyancy effect;
- An average interparticle stress which in accordance with continuum mechanics acts through the total volume and is given by Terzaghi's effective stress σ'_{ij}.

Thus the total stress may be written as,

$$\sigma_{ij} = (nu_w + (1\text{-}n)u_w)\delta_{ij} + \sigma'_{ij} \qquad (6)$$

This correctly reduces to Terzaghi's effective stress Equation (5). The above arguments for a saturated soil may be extended to an unsaturated soil in the following way.

3. UNSATURATED SOIL

In an equivalent 'energy formulation' for unsaturated soil, it is necessary to take account of the following:

- The water pressure acting through the volume of water;
- The air pressure acting through the volume of air;
- A fluid pressure acting through the volume of the solid phase as the voids are filled with air and water;
- An average interparticle stress which in accordance with the principles of continuum mechanics may be considered as acting through the total volume and is denoted by the stress tensor $\sigma'_{c,ij}$.
- The interaction effects between the three phases - surface tension due to the contractile skin; the adsorbed double layer of water on the soil particles; the dissolved air; and the water vapour.

Murray (2002) has shown from comparison with experimental data on kaolin (Wheeler and

Sivakumar, 1995 and 2000) that the interaction effects can be ignored and the fluid pressure through the soil particles may be taken as the water pressure. This latter simplification is consistent with the water having a close affinity to the soil particles and with the concept of saturated 'packets' comprising the solid particles and the water, surrounded by air filled voids. Thus the total stress may be written as,

$$\sigma_{ij} = [n_w u_w + (1\text{-}n) u_w + n_a u_a]\delta_{ij} + \sigma'_{c,ij} \qquad (7)$$

where, $\sigma'_{c,ij}$ is defined as the 'coupling stress' tensor; n_w is the volume of water per unit volume of soil; and n_a is the volume of air per unit volume of soil.

Equation (7) may be written in terms of stress state variables as,

$$\sigma'_{c,ij} = (\sigma_{ij} - u_a\, \delta_{ij}) + [s\, v_w/v]\, \delta_{ij} \qquad (8)$$

where, $s = (u_a\text{-}u_w)$; v is the specific volume; and v_w is the specific water volume

Under triaxial stress conditions, it is consistent with the normal definition for mean stress that $p = (\sigma_1 + 2\sigma_3)/3$ and $p'_c = (\sigma'_{c,1} + 2\sigma'_{c,3})/3$, thus,

$$p'_c = (p - u_a) + s\, v_w/v \qquad (9)$$

where, p is the mean total stress; and p'_c is the average volumetric coupling stress

4. CONJUGACY AND WORK INPUT

Murray (2002) derives Equation (9) from considerations of enthalpy and defines p'_c as the *'average volumetric coupling stress'* as it links the stress state variables $(p\text{–}u_a)$ and $(u_a\text{-}u_w)$ to the volumes of the phases represented by v_w/v. The ratio v_w/v is the volume of the saturated packets (or regions) per unit volume of soil. The equation contains volumetric terms for all three phases. In accordance with Equation (9), $(p\text{–}u_a)$ may be thought of as acting through the whole of the soil mass with the additional pressure $(u_a\text{-}u_w)$ acting within the saturated packets. This defines the dual stress regime considered to exist in unsaturated soils as a result of a bi-modal structure, as well as defining the volumetric variables conjugate to the stress state variables. Equation (9) has been shown (Murray, 2002; Murray et al., 2002) to provide a clear and logical picture of the behaviour of unsaturated soils tested in the triaxial cell.

Equations (8) and (9) reflect the fact that σ'_c and p'_c are conjugate to v, $(\sigma - u_a)$ and $(p - u_a)$ are conjugate to v and $(u_a\text{-}u_w)$ is conjugate to v_w. Conjugate in this sense does not refer to work conjugate but describes the volumes through which the stress state variables act. While saturated soils are amenable to analysis using the principles of continuum mechanics, such an approach is not as clearly applicable in the analysis of unsaturated soils. In saturated soils the single stress state variable controlling the behaviour is Terzaghi's effective stress. In unsaturated soils any two of the three stress state variables may be used as the third stress state variable may be determine from the other two and accordingly Equations (8) and (9) may be written in any of three forms (Murray et al., 2002). The stress state variables act within clearly defined volumes in accordance with these equations. If continuum mechanics is to be invoked for unsaturated soils it must be viewed in terms of the duality of the stress regime and the volumes through which the stresses act.

The increment of work input dW associated with the stress state variables must take account of the volumes through which the stresses act. Accordingly, the work input may be written as,

$$dW = (\sigma_{ij} - u_a)\, d\varepsilon_{ij} + s\, d\varepsilon_{w,ij} \qquad (10)$$

or for the triaxial test,

$$dW = (p - u_a)\, d\varepsilon_v + s\, d\varepsilon_w + q\, d\varepsilon_s \qquad (11)$$

where, $d\varepsilon_{ij}$ is the strain-increment tensor for the whole soil mass; $d\varepsilon_{w,ij}$ is the strain-increment tensor for the saturated packets; $d\varepsilon_v$ is the volumetric strain-increment for the whole soil mass; $d\varepsilon_w$ is the volumetric strain-increment for the saturated packets (dv_w/v); and $d\varepsilon_s$ is the shear strain-increment for the whole soil mass. For a saturated soil $d\varepsilon_w = d\varepsilon_v$ and equation (11) reduces to $dW = (p\text{–}u_w)d\varepsilon_v + q\, d\varepsilon_s$. In accordance with Equation (10), $(\sigma_{ij} - u_a)$ is work conjugate to $d\varepsilon_{ij}$ and s conjugate to $d\varepsilon_{w,ij}$. In accordance with Equation (11), $(p - u_a)$ is conjugate to $d\varepsilon_v$, s is conjugate to $d\varepsilon_w$ and q is conjugate to $d\varepsilon_s$.

5. ANALYSIS OF TRIAXIAL TEST DATA

Figure 2 presents plots of conjugate stress and strain variables based on equation (11) for the shearing stage of a triaxial stress test on kaolin after Wheeler and Sivakumar (1995). In the test s was held constant at 300kPa, while air and water drainage were allowed. The specimen may be considered as normally consolidated. For full details of the specimen preparation and test protocols the original paper should be consulted. The strain increments are the cumulative increments during the shearing stage of the test. Included is a plot of $p\text{-}u_w$ against $d\varepsilon_w$; the vertical difference between this plot and the plot of $p\text{-}u_a$ against $d\varepsilon_v$ being constant at s=300kPa. The plots indicate consistent trends with overall volumetric compression of the specimen indicated by the positive values of $d\varepsilon_v$ but expansion of the saturated regions consistent with the increase in water content during shearing as indicated by the negative values of dε_w. The air void space is thus greatly reduced in the test.

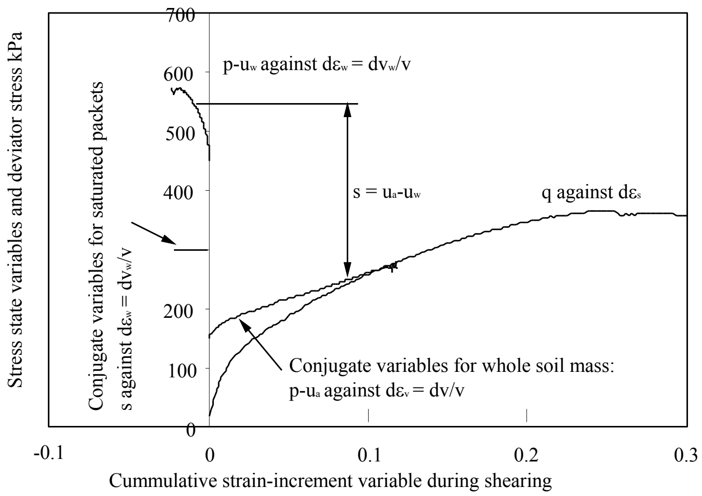

Figure 2 Stress state variables and q against cumulative strain-increment variables

6. CONCLUSIONS

An equation is developed for the stress regime in an unsaturated soil using an 'energy formulation'. The stress regime is described in terms of stress tensors and triaxial stress conditions using two of the stress state variables and the volumes through which the stresses act. The equations have been used to develop the work input equations associated with the stress state variables and which illustrate the work conjugate strain-increments appropriate in analyzing experimental data and developing a stress-strain model for unsaturated soils using constitutive relationships. The use of the conjugate variables is illustrated by plotting the results for the shearing stage of a triaxial compression test.

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