Equilibrium stress conditions in unsaturated soil

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Abstract

The significance of the thermodynamic potential and extensive variables is outlined and leads to identification of the role and use of enthalpy in a multi-phase material comprising solid particles, liquid and gas. This is presented in terms of triaxial stress conditions. Enthalpy considerations are then used to develop equations that describe the equilibrium stress regime in an unsaturated soil and demonstrate the dependence on the relative volumes of the phases. The analysis supports and expands on previously published work.

Introduction

Bishop (1959) proposed a single-stress variable σ'_B as the controlling stress in unsaturated soils. This incorporates a parameter χ to account for the effect of matric suction $(u_a\text{-}u_w)$ on mechanical behaviour in unsaturated soil.

[1] $$\sigma'_B = (\sigma - u_a) + \chi(u_a - u_w)$$

where, σ is total stress
u_a is pore air pressure
u_w is pore water pressure
χ is assumed to be a function of degree of saturation.

This may alternatively be written in terms of mean stress under 'triaxial' test conditions as,

[2] $$p'_B = (p - u_a) + \chi(u_a\text{-}u_w)$$

where, p is mean total stress = $(\sigma_1+2\sigma_3)/3$
p'_B is mean Bishop's stress = $(\sigma'_{1B}+2\sigma'_{3B})/3$
σ_1 and σ_3 are the total axial and lateral principal stresses respectively
σ'_{1B} and σ'_{3B} are the Bishop axial and lateral stresses respectively

However, it is widely appreciated that experimental evidence suggests that the use of a single stress variable such as σ'_B is incapable of adequately describing the stress regime in an unsaturated soil (e.g. Jennings and Burland 1962; Morgenstern 1979). As a result, researchers have turned to 'independent stress state variables' to describe the strength and volume change characteristics of unsaturated soils (e.g. Matyas and Radhakrishna 1968; Fredlund and Morgenstern 1977). This approach suggests that any two of the three stress state variables $(\sigma\text{-}u_a)$, $(\sigma\text{-}u_w)$ and $(u_a\text{-}u_w)$ may be used in describing the mechanical behaviour of an unsaturated soil.

Nevertheless, theoretical analyses yield equations similar in form to Equations [1] and [2]. Houlsby (1997) examined the power input into unsaturated soils and developed a relationship where χ is replaced with the degree of saturation S_r. Li (2003) presented a microstructural analysis and concluded that there is a quasi-effective stress for unsaturated soils with χ replaced by a tensor term describing the distribution of pore pressures. Murray (2002) derived an equation to describe the general equilibrium stress conditions in unsaturated soils that may be written as,

[3] $p'_c = (p - u_a) + (u_a - u_w)\, v_w/v$

where, p'_c is defined as the mean coupling stress
v is the specific volume given by $(1+e)$
v_w is the specific water volume given by $(1+e\,S_r)$
e is the void ratio

The equation was developed from considerations of enthalpy and although in a form similar to Bishop's Equation [2] should not be confused with it as the volumetric terms indicate a dual stress regime not a single stress as controlling the behaviour of unsaturated soils. The equation has proved insightful in analysing published experimental data on unsaturated soils under equilibrium conditions (Murray, 2002; Murray et al, 2002; Murray and Sivakumar, 2004). An alternative derivation of Equation [3] is presented. First the significance of the thermodynamic extensive and intensive variables is discussed.

Thermodynamic extensive and intensive variables

The thermodynamic concepts of internal energy and entropy may without modification be applied to soils. Under equilibrium conditions assuming no chemical potential, thus no osmotic suction, and ignoring the gravitational field, the internal energy U for an isotropically loaded specimen of volume V may be written as,

[4] $U = TS - pV$

where, T is the absolute temperature of the specimen
S is the entropy of the specimen
p is the applied pressure

Under conditions of constant T and p, for any subsequent change in the variables of state S and V, the internal energy given by Equation [1] represents the Euler thermodynamic potential (Sposito, 1981). In applying this to soils in a triaxial cell no distinction is made as to the degree of saturation or composition of the specimen. Thus only the applied total stresses are considered and Equation [4] does not consider internal stresses or fluid pressures in the soil specimen and says nothing about the specimen's history. Essentially an undrained test under closed conditions with no matter exchange with the triaxial measuring or pressure system is allowed.

For anisotropic triaxial loading of a cylindrical soil specimen of height h and radius r the change in internal energy for reversible, infinitesimal (virtual) changes in thermal and mechanical energy as a result of perturbations dS and dV respectively, is given by,

$$[5] \qquad dU = TdS - p_m\,dV - q_m \frac{2}{3} V \left(\frac{dh}{h} - \frac{dr}{r}\right)$$

where, $p_m = (\sigma_1 + 2\sigma_3)/3$

$q_m = (\sigma_1 - \sigma_3)$.

dV is the volume change of the specimen

dh is the axial compression of the soil specimen

dr is the radial compression of the specimen

The thermodynamic potential U for anisotropic loading conditions may be determined from Equation [5] and is given by,

$$[6] \qquad U = TS - p_m\,V$$

Equation [6] is in the same form as the Euler Equation [4] for isotropic loading but p_m represents the mean stress. There is no separate term for the deviator stress q_m as on integration of Equation [5] the conjugate deviator strain term reduces to zero. The fact that a separate term for q_m does not appear in Equation [5] indicates that at equilibrium it is appropriate to treat p_m and q_m independently. For simplicity the suffix 'm' will be dispensed with in the following.

It is important to distinguish between the extensive variables and intensive variables used in the analysis. The extensive variables are dependent on the amount of matter in the soil, whereas the values of the intensive variables can vary from point to point. The distinction between the variables is significant as the extensive variables are 'additive'. The extensive variables include U, V and S, and, for example, the total volume of the phases in a soil is the sum of the volumes of the individual phases. The intensive variables comprise p and T and it is inappropriate to sum the values for the individual phases. However, the product of an extensive variable and an intensive variable such as pV is also an extensive variable and is additive as is the influence of the individual stresses making up the product. The terms in the thermodynamic potential Equation [6] are thus extensive variables as is the thermodynamic potential itself. The enthalpy defined as H = pV+U can thus also be seen to be an extensive variable (Callen, 1965; Sposito, 1981).

Role of Enthalpy

Murray (2002) developed an analysis using enthalpy that allows an equation to be formulated describing the equilibrium stress regime in unsaturated soils and the following expands on the major considerations of the analysis. First however the principles on which the analysis depends are outlined.

Principle 1 - The total enthalpy H of a multi-phase material is the sum of the enthalpies of the individual phases and the interactions between the phases H_i.
As enthalpy is an extensive variable, for a material comprising a number of phases, the total enthalpy comprises the summed enthalpies of the phases and their interactions. This also holds true for unsaturated soils. The interactions considered comprise: water vapour in air; dissolved air in water; the contractile skin between water and air; adsorbed water on the soil particles; and soil particle interactions. While in most cases the product pV may be used within the enthalpy equation for the phase or interaction, it is important to note that for the contractile skin it is necessary to replace this with the product of the surface tension and the area of the film.

Interactions may be miscible or immiscible. A miscible interaction such as water vapour in air may be taken as obeying Dalton's divisional law of partial pressures whereas an immiscible interaction, such as the adsorbed water on soil particles, may be treated as having its own pressure acting through its own volume. Let p_i be the pressure or spherical stress component arising from a phase or interaction between phases. For those miscible interactions obeying Dalton's divisional law, the partial pressures p_p may all be taken to act through the total volume of the phase. Accordingly the total pressure p_i for a miscible interaction is given by $p_i = \Sigma p_p$. Thus the total pressure p for a multi-phase material with both immiscible and miscible interactions, assuming a surface encompassing the soil specimen that passes through a representative volume of homogeneous soil, is the sum of the pressures of the phases and their interactions factored by the relative volumes V_i/V giving $p = \Sigma p_i V_i/V$. Further, since U is an extensive variable the total U may be equated to the sum of the internal energies of the phases and their interactions ($U=\Sigma Ui$) and from the definition of enthalpy,

[7] $H = \Sigma Hi$

This holds true for isotropic and anisotropic loading as the influence of the individual stresses are additive. Accordingly, the total enthalpy of an unsaturated soil specimen may be determined from summation of the enthalpies of the individual phases and their interactions.

Principle 2 - The sum of the enthalpies of the individual phases in isolation equals the enthalpy of the combined phases if no work is done in combining the phases.
It is necessary to consider a system where there is no mass loss or heat exchange with the surroundings. This idealised system is necessary and appropriate as no actual process of combining the phases is undertaken but only a theoretical redistribution of the phases within a given volume is considered. The general work equation may be written as,

[8] $dU = dQ + dW = TdS - pdV$
where, $dQ = TdS$ is the change in heat of the soil
$dW = pdV$ is the work done on the soil

This assumes that the combined phases experience isotropic loading conditions and as $dQ = 0$ this leads to $dU = dW = -pdV$. If the process is one where there is no volume change as the phases are combined, then no net work is done and $dW = dU = dH = 0$. Thus in combining incompressible phases, equilibrium conditions require that,

[9] $dH = \Sigma dHi = 0$

where, Hi represents an individual phase

Equation [9] states that there is no change in total enthalpy, thus the sum of the enthalpies of the individual phases in isolation equates to the sum of the enthalpies of the combined phases. As water and soil particles are usually considered incompressible, Equation [9] is considered to apply to the idealised unsaturated soil structure comprising ‘saturated packets’ surrounded by air voids. The saturated packets are created as a result of the difference between the air and water pressure and thus arise from isotropic loading conditions.

Analysis using Enthalpy

The foregoing has set the background to the analysis that follows which first considers a simple fluid, then a gaseous fluid before extending the arguments to an unsaturated soil comprising the three phases of solid particles, liquid and gas.

Analysis of an element of de-aired water: Figure 1 shows a small element of de-aired water of volume $V=V_w$ in equilibrium under an isotropic pressure p. The weight of the element is taken as negligible. The external pressure is balanced by the internal water pressure u_w. To an external observer the enthalpy of the element $H = pV + U$. To an internal observer the enthalpy would be $H = u_wV_w+U$. Internal to the element refers to the recognition that there is a water pressure resisting the applied pressure p. The simple conclusion is that $p = u_w$ irrespective of the value of U.

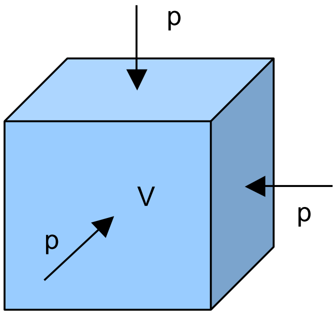

Figure 1: Element of de-aired water

Analysis of an element of water with air bubbles: To an external observer the enthalpy of the element depicted in Figure 2 is again given by $H = pV+U$. However, internal to the system, an observer would give $H=u_wV_w+u_aV_a+T+D+W+U$ where U is the sum of the internal energies associated with the air and water phases. T represents the enthalpy associated with the surface tension, D represents the enthalpy associated with the dissolved air and W represents the enthalpy associated with the water vapour in the air phase.

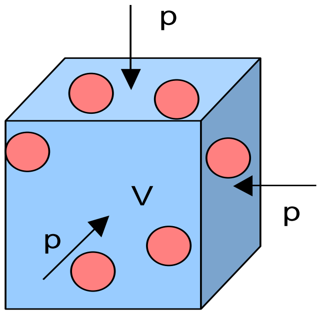

Figure 2: Element of water with air bubbles

The influences of the dissolved air and water vapor are small and may be ignored but are included for completeness. The influence of the surface tension T may also be small but plays a far more significant role as the relative volume of air to water increases. Bearing this in mind, under equilibrium conditions,

[10] $pV = u_wV_w + u_aV_a + T + D + W$

Analysis of an element of water with air and interacting soil particles

To an external observer the enthalpy is again given by $H = pV+U$. Internal to the system however, $H = u_wV_w+u_aV_a+u_sV_s+p'_cV+T+D+A+U$. The 'effective' stress p'_c is the overall effect in terms of continuum mechanics of the inter-particle forces. There is a jump in philosophy from enthalpy to continuum mechanics but a necessary one as the actual interaction stresses between particles are indeterminate. p'_c acts through the total volume V. Thus,

[11] $pV = u_wV_w+u_aV_a+u_sV_s+p'_cV+T+D+A$

The analysis of Murray (2002) shows that D and A can be neglected as they represent only small components of enthalpy, giving,

[12] $p'_c = (p-u_a)+\alpha(u_a-u_w)(n_w+n_s) = (p-u_a)+\alpha(u_a-u_w)v_w/v$

where, n_w is the volume of water per unit volume of soil
n_s is the volume of solids per unit volume of soil
$v_w/v=(V_w+V_s)/V$ is the volume of saturated ‘packets’ per unit volume of soil.

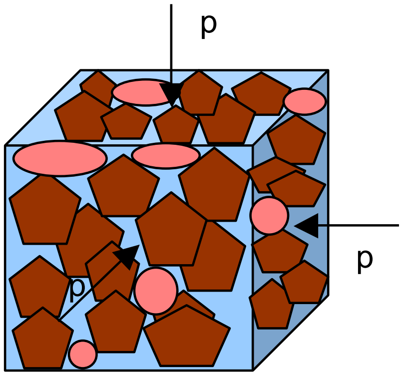

Figure 3: Element of water with air bubbles and interacting soil particles

Murray (2002) also shows that the term α is given by,

$$[13] \quad \alpha = 1-b-(a-1)\,\frac{n_s}{(n_w+n_s)}\,\frac{u_w}{(u_a-u_w)}$$

The dimensionless number ‘a’ relates to the pressure in the soil particles and the dimensionless parameter ‘b’ relates to the influence of the contractile skin and the size and shape of the saturated packets.

Analysis using the visual idealisation of saturated packets assuming the water phase and all soil particles are contained within the packets gives a further equation for α, Equation [14] (Murray, 2002). The idealisation allows a visualisation of the mathematics involved but should not be taken as restricting the use of the equations developed to a limited range of soil suctions and degrees of saturation. The only stipulations in the analysis undertaken is that the soil particles and water are intrinsically linked either by water surrounding the particles or by water bridges at points of particle contact, and that at equilibrium the suction is everywhere the same. It is recognised that the idealisation may not be perceived as representing the conditions particularly near the extremes of either a near-saturated soil or a very dry soil, and particularly where the soil includes or comprises coarse particles such as sand. However, the equations developed are thought to apply to a wide range of conditions and soils, though only further experimental evidence will fully justify the range of their usage. Equation [14] is based on Principle 2 developed in the foregoing.

$$[14] \quad \alpha = 1 + \frac{5(u_a-u^*_w)\,n_w}{2(u_a-u_w)(n_w+n_s)}$$

where, $(u_a-u^*_w)$ is negative being the difference between the external air pressure and the water pressure inside a spherical bubble of water with the same volume as that of the water within a saturated packet.

For a dry soil $n_w \approx 0$ and Equation [14] gives $\alpha \approx 1$. For a saturated or near-saturated soil, if Terzaghi’s effective stress equation applies, in accordance with Equation [13]

$(n_w+n_s)=1$ and again $\alpha = 1$. Between these extremes α is ≤ 1. However, comparison with the experimental evidence of Wheeler and Sivakumar (1995) suggests that the equality holds very closely true at intermediate degrees of saturation. Published experimental evidence for a range of materials also suggests the applicability of the above with $\alpha=1$. Thus Equation [12] becomes,

[15] $p'_c=(p-u_a)+(u_a-u_w)v_w/v$

As discussed previously, on the basis that enthalpy is an extensive variable, the influence of the individual stresses is additive and from Equation [15] it is possible to write the general stress tensor equation for unsaturated soils under equilibrium conditions.

[16] $\sigma'_{c,ij} = (\sigma_{ij}-u_a\delta_{ij})+(u_a-u_w)\delta_{ij}v_w/v$

where, $\sigma'_{c,ij}$ is the 'coupling stress' tensor

σ_{ij} is the total stress tensor

δ_{ij} is the Kronecker delta

Conclusions

The significance of enthalpy as a thermodynamic extensive variable has been discussed and has been used to determine the stress regime in unsaturated soils comprising the interacting phases of solid particles, water and air.

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